The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING

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Topic Generator - Solution Set
Solutions

1. The value of $8+2(3^2)$ is
   (A) 26 (B) 90 (C) 41 (D) 44 (E) 60

Source: 2007 Cayley Grade 10 #1

Primary Topics: Number Sense

Secondary Topics: Operations

Answer: A

Solution:

Calculating, $8+2(3^2)=8+2(9)=8+18=26$.

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2. What is the value of $(-1)^5-(-1)^4$?
   (A) -2 (B) -1 (C) 0 (D) 1 (E) 2

Source: 2009 Pascal Grade 9 #6

Primary Topics: Number Sense

Secondary Topics: Exponents

Answer: A

Solution:

When -1 is raised to an even exponent, the result is 1.
When -1 is raised to an odd exponent, the result is -1.
Thus, $(-1)^5-(-1)^4=-1-1=-2$.

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3. Which is the largest sum?
   (A) $\frac{1}{4}+\frac{1}{5}$ (B) $\frac{1}{4}+\frac{1}{6}$ (C) $\frac{1}{4}+\frac{1}{3}$ (D) $\frac{1}{4}+\frac{1}{8}$ (E) $\frac{1}{4}+\frac{1}{7}$

Source: 2010 Gauss Grade 8 #6

Primary Topics: Number Sense

Secondary Topics: Fractions/Ratios | Optimization | Operations

Answer: C

Solution:

We first recognize that $\frac{1}{4}$ is a common fraction in each of the five sums, and so the relative size of the sums depends only on the other fractions.
Since $\frac{1}{3}$ is the largest of the fractions $\frac{1}{5}, \frac{1}{6}, \frac{1}{3}, \frac{1}{8}, \frac{1}{7}$, we conclude that $\frac{1}{4}+\frac{1}{3}$ is the largest sum.

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4. The value of $10^2 + 10 + 1$ is
   (A) 101 (B) 1035 (C) 1011 (D) 111 (E) 31

Source: 2013 Gauss Grade 8 #1

Primary Topics: Number Sense

Secondary Topics: Operations | Exponents

Answer: D

Solution:

Evaluating, $10^2 + 10 + 1 = 10 \times 10 + 10 + 1 = 100 + 10 + 1 = 111$.

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5. If there is no tax, which of the following costs more than $18 to purchase?
   (A) Five $1 items and five $2 items
   (B) Nine $1 items and four $2 items
   (C) Nine $1 items and five $2 items
   (D) Two $1 items and six $2 items
   (E) Sixteen $1 items and no $2 items

Source: 2018 Gauss Grade 7 #5

Primary Topics: Number Sense

Secondary Topics: Operations

Answer: C

Solution:

The cost of nine $1 items and five $2 items is $(9 \times \$1) + (5 \times \$2)$, which is $\$9 + \$10$ or $19.
The correct answer is (C).
(We may check that each of the remaining four answers gives a cost that is less than $18.)

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6. An electric car is charged 3 times per week for 52 weeks. The cost to charge the car each time is $0.78. What is the total cost to charge the car over these 52 weeks?
   (A) $104.00 (B) $81.12 (C) $202.80 (D) $162.24 (E) $121.68

Source: 2018 Pascal Grade 9 #4

Primary Topics: Number Sense

Secondary Topics: Operations

Answer: E

Solution:

Since the car is charged 3 times per week for 52 weeks, it is charged $3 \times 52 = 156$ times.
Since the cost per charge is $0.78, then the total cost is $156 \times \$0.78 = \$121.68$.

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7. The expression $2 \times 0 + 1 - 9$ equals
   (A) -8 (B) -6 (C) -7 (D) -11 (E) 0

Source: 2019 Cayley Grade 10 #1

Primary Topics: Number Sense

Secondary Topics: Operations

Answer: A

Solution:

Evaluating, $2 \times 0 + 1 - 9 = 0 + 1 - 9 = -8$.

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8. In a sequence of numbers, the first term is 3. Each new term is obtained by adding 5 to the previous term. The first four terms are $3, 8, 13, 18$. What are the next three terms in the sequence?
   (A) $25, 30, 35$ (B) $5, 10, 15$ (C) $23, 28, 33$ (D) $23, 33, 43$ (E) $19, 20, 21$

Source: 2022 Gauss Grade 7 #4

Primary Topics: Algebra and Equations

Secondary Topics: Patterning/Sequences/Series | Operations

Answer: C

Solution:

Since $18 + 5 = 23$, and $23 + 5 = 28$, and $28 + 5 = 33$, the next three terms in the sequence are $23, 28, 33$.

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9. A mother bear collects 14 fish. She gives 4 fish to each of her 3 bear cubs. How many fish does the mother bear have left over?
   (A) 0 (B) 2 (C) 3 (D) 4 (E) 5

Source: 2025 Cayley Grade 10 #2

Primary Topics: Number Sense

Secondary Topics: Counting | Operations

Answer: B

Solution:

Mother bear gives a total of $4 \times 3 = 12$ fish to her bear cubs, and so she has $14 - 12 = 2$ fish left over.

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10. If $50-2\sqrt{x}=18$, the value of x is
(A) 32 (B) 16 (C) 64 (D) 256 (E) 8

Source: 2025 Cayley Grade 10 #7

Primary Topics: Algebra and Equations

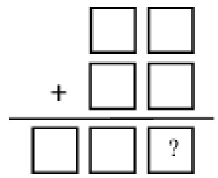
Secondary Topics: Equations Solving | Operations

Answer: D

Solution:

Solving $50-2\sqrt{x}=18$, we get $-2\sqrt{x}=18-50$ or $-2\sqrt{x}=-32$ or $\sqrt{x}=16$, and so $x=16^2=256$.

11. In the addition of two 2-digit numbers, each blank space, including those in the answer, is to be filled with one of the digits 0, 1, 2, 3, 4, 5, 6, each used exactly once. The units digit of the sum is



(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Source: 2006 Gauss Grade 8 #20

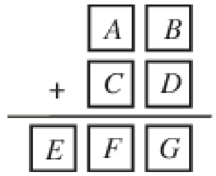
Primary Topics: Number Sense

Secondary Topics: Digits

Answer: D

Solution:

We label the blank spaces to make them easier to refer to.

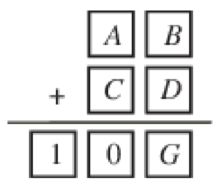


Since we are adding two 2-digit numbers, then their sum cannot be 200 or greater, so E must be a 1 if the sum is to have 3 digits.

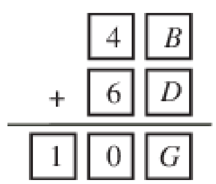
Where can the digit 0 go?
Since no number can begin with a 0, then neither A nor C can be 0.
Since each digit is different, then neither B and D can be 0, otherwise both D and G or B and G would be the same.
Therefore, only F or G could be 0.

Since we are adding two 2-digit numbers and getting a number which is at least 100, then $A+C$ must be at least 9. (It could be 9 if there was a ``carry" from the sum of the units digits.)
This tells us that A and C must be 3 and 6, 4 and 5, 4 and 6, or 5 and 6.

If G was 0, then B and D would have to 4 and 6 in some order. But then the largest that A and C could be would be 3 and 5, which are not among the possibilities above.
Therefore, G is not 0, so $F=0$.

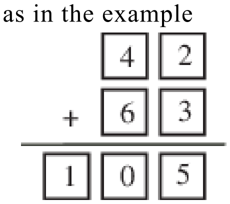


So the sum of A and C is either 9 or 10, so A and C are 3 and 6, 4 and 5, or 4 and 6.
In any of these cases, the remaining possibilities for B and D are too small to give a carry from the units column to the tens column.
So in fact, A and C must add to 10, so A and C are 4 and 6 in some order.
Let's try $A=4$ and $C=6$.



The remaining digits are 2, 3 and 5. To make the addition work, B and D must be 2 and 3 and G must be 5. (We can check that either order for B and D works, and that switching the 4 and 6 will also work.)

So the units digit of the sum must be 5, as in the example



(Note that we could have come up with this answer by trial and error instead of this logical procedure.)

12. If x and y are two-digit positive integers with $xy = 555$, what is $x + y$?
(A) 52 (B) 116 (C) 66 (D) 555 (E) 45

Source: 2008 Cayley Grade 10 #15

Primary Topics: Algebra and Equations

Secondary Topics: Divisibility

Answer: A

Solution:

First, we find the prime factors of 555.
Since 555 ends with a 5, it is divisible by 5, with $555 = 5 \times 111$.
Since the sum of the digits of 111 is 3, then 111 is divisible by 3, with $111 = 3 \times 37$.
Therefore, $555 = 3 \times 5 \times 37$, and each of 3, 5 and 37 is a prime number.
The possible ways to write 555 as the product of two integers are 1×555, 3×185, 5×111, and 15×37. (In each of these products, two or more of the prime factors have been combined to give a composite divisor.)
The only pair where both members are two-digit positive integers is 37 and 15, so $x + y$ is $37 + 15 = 52$.

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13. The sum of four numbers is T. Suppose that each of the four numbers is now increased by 1. These four new numbers are added together and then the sum is tripled. What is the value of this final result?
(A) $3T + 3$ (B) $3T + 4$ (C) $3T + 12$ (D) $T + 12$ (E) $12T$

Source: 2011 Gauss Grade 7 #19

Primary Topics: Algebra and Equations

Secondary Topics: Operations

Answer: C

Solution:

If each of the four numbers is increased by 1, then the increase in their sum is 4.
That is, these four new numbers when added together have a sum that is 4 more than their previous sum T, or $T + 4$.
This new sum $T + 4$ is now tripled.
The result is $3 \times (T + 4) = (T + 4) + (T + 4) + (T + 4)$ or $3T + 12$.

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14. The operation ∇ is defined by $g\nabla h = g^2 - h^2$. For example, $2\nabla 1 = 2^2 - 1^2 = 3$. If $g > 0$ and $g\nabla 6 = 45$, the value of g is

(A) 39 (B) 6 (C) 81 (D) 3 (E) 9

Source: 2012 Pascal Grade 9 #13

Primary Topics: Algebra and Equations

Secondary Topics: Equations Solving | Operations

Answer: E

Solution:

Using the definition of the operation, $g\nabla 6 = 45$ gives $g^2 - 6^2 = 45$.
Thus, $g^2 = 45 + 36 = 81$.
Since $g > 0$, then $g = \sqrt{81} = 9$.

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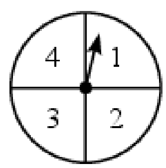
15. On each spin of the spinner shown, the arrow is equally likely to stop on any one of the four numbers. Deanna spins the arrow on the spinner twice. She multiplies together the two numbers on which the arrow stops. Which product is most likely to occur?



(A) 2 (B) 4 (C) 6 (D) 8 (E) 12

Source: 2014 Pascal Grade 9 #19

Primary Topics: Number Sense | Counting and Probability

Secondary Topics: Probability | Prime Numbers | Factoring | Divisibility

Answer: B

Solution:

We make a chart that lists the possible results for the first spin down the left side, the possible results for the second spin across the top, and the product of the two results in the corresponding cells:
Since each spin is equally likely to stop on 1, 2, 3, or 4, then each of the 16 products shown in the chart is equally likely.
Since the product 4 appears three times in the table and this is more than any of the other numbers, then it is the product that is most likely to occur.

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16. The operation $\otimes$ is defined by $a \otimes b = \dfrac{a}{b} + \dfrac{b}{a}$. What is the value of $4 \otimes 8$?

(A) $\frac{1}{2}$ (B) 1 (C) $\frac{5}{4}$ (D) 2 (E) $\frac{5}{2}$

Source: 2015 Cayley Grade 10 #11

Primary Topics: Number Sense

Secondary Topics: Operations

Answer: E

Solution:

From the given definition,

$$4 \otimes 8 = \frac{4}{8} + \frac{8}{4} = \frac{1}{2} + 2 = 2\frac{1}{2} = \frac{5}{2}$$

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17. In the sum shown, P and Q each represent a digit. The value of $P + Q$ is

$$\begin{array}{r} PQQ \\ PPQ \\ +\,QQQ \\ \hline 876 \end{array}$$

(A) 3 (B) 5 (C) 7 (D) 6 (E) 4

Source: 2016 Gauss Grade 7 #12

Primary Topics: Number Sense

Secondary Topics: Digits | Operations

Answer: B

Solution:

The sum of the units column is $Q + Q + Q = 3Q$.
Since Q is a single digit, and $3Q$ ends in a 6, then the only possibility is $Q = 2$.
Then $3Q = 3 \times 2 = 6$, and thus there is no carry over to the tens column.
The sum of the tens column becomes $2 + P + 2 = P + 4$, since $Q = 2$.
Since P is a single digit, and $P + 4$ ends in a 7, then the only possibility is $P = 3$.
Then $P + 4 = 3 + 4 = 7$, and thus there is no carry over to the hundreds column.
We may verify that the sum of the hundreds column is $3 + 3 + 2 = 8$, since $P = 3$ and $Q = 2$.
The value of $P + Q$ is $3 + 2 = 5$, and the final sum is shown.

$$\begin{array}{r} 322 \\ 332 \\ +\,222 \\ \hline 876 \end{array}$$

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18. The mean (average) height of a group of children would be increased by 6 cm if 12 of the children in the group were each 8 cm taller. How many children are in the group?
(A) 16 (B) 14 (C) 21 (D) 26 (E) 9

Source: 2018 Gauss Grade 8 #19

Primary Topics: Data Analysis | Algebra and Equations

Secondary Topics: Equations Solving | Averages

Answer: A

Solution:

The mean height of the group of children is equal to the sum of the heights of the children divided by the number of children in the group.
Therefore, the mean height of the group of children increases by 6 cm if the sum of the increases in the heights of the children, divided by the number of children in the group, is equal to 6.
If 12 of the children were each 8 cm taller, then the sum of the increases in the heights of the children would be $12 \times 8 = 96$ cm.
Thus, 96 divided by the number of children in the group is equal to 6.
Since $96 \div 16 = 6$, then the number of children in the group is 16.

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19. Janet picked a number, added 7 to the number, multiplied the sum by 2, and then subtracted 4. If the final result was 28, what number did Janet pick?
(A) 9 (B) 5 (C) 19 (D) 23 (E) 11

Source: 2017 Pascal Grade 9 #13

Primary Topics: Number Sense | Algebra and Equations

Secondary Topics: Operations

Answer: A

Solution:

Solution 1
We undo Janet's steps to find the initial number.
To do this, we start with 28, add 4 (to get 32), then divide the sum by 2 (to get 16), then subtract 7 (to get 9).
Thus, Janet's initial number was 9.

Solution 2
Let Janet's initial number be x.
When she added 7 to her initial number, she obtained $x + 7$.
When she multiplied this sum by 2, she obtained $2(x + 7)$ which equals $2x + 14$.
When she subtracted 4 from this result, she obtained $(2x + 14) - 4$ which equals $2x + 10$.
Since her final result was 28, then $2x + 10 = 28$ or $2x = 18$ and so $x = 9$.

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20. Mateo’s 300 km trip from Edmonton to Calgary passed through Red Deer. Mateo started in Edmonton at 7 a.m. and drove until stopping for a 40 minute break in Red Deer. Mateo arrived in Calgary at 11 a.m. Not including the break, what was his average speed for the trip?

(A) 83 km/h (B) 94 km/h (C) 90 km/h (D) 95 km/h (E) 64 km/h

Source: 2022 Gauss Grade 8 #19

Primary Topics: Geometry and Measurement

Secondary Topics: Rates | Equations Solving | Averages

Answer: C

Solution:

There are 60 minutes in an hour, and so the number of minutes between 7 a.m. and 11 a.m. is $4 \times 60 = 240$.

Since Mateo stopped for a 40 minute break, he drove for $240 - 40 = 200$ minutes.

Thus, the average speed for Mateo’s 300 km trip was $\dfrac{300\text{ km}}{200\text{ minutes}} = 1.5\text{ km/min}$.

Since there are 60 minutes in an hour, if Mateo averaged 1.5 km per minute, then he averaged 1.5×60 km per hour or 90 km/h.

21. A positive integer is called a *perfect power* if it can be written in the form a^b, where a and b are positive integers with $b \geq 2$. For example, 32 and 125 are perfect powers because $32 = 2^5$ and $125 = 5^3$. The increasing sequence

$$2, 3, 5, 6, 7, 10, \ldots$$

consists of all positive integers which are not perfect powers. The sum of the squares of the digits of the 1000th number in this sequence is

(A) 42 (B) 26 (C) 33 (D) 18 (E) 21

Source: 2005 Pascal Grade 9 #24

Primary Topics: Counting and Probability

Secondary Topics: Patterning/Sequences/Series | Exponents

Answer: E

Solution:

Since we have an increasing sequence of integers, the 1000th term will be at least 1000.
We start by determining the number of perfect powers less than or equal to 1000. (This will tell us how many integers less than 1000 are "skipped" by the sequence.)
Let us do this by making a list of all perfect powers less than 1100 (we will need to know the locations of some perfect powers bigger than 1000 anyways):

- Perfect squares: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900, 961, 1024, 1089
- Perfect cubes: 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000
- Perfect 4th powers: 1, 16, 81, 256, 625
- Perfect 5th powers: 1, 32, 243, 1024
- Perfect 6th powers: 1, 64, 729
- Perfect 7th powers: 1, 128
- Perfect 8th powers: 1, 256
- Perfect 9th powers: 1, 512
- Perfect 10th powers: 1, 1024

In these lists, there are 41 distinct perfect powers less than or equal to 1000.
Thus, there are 959 positive integers less than or equal to 1000 which are not perfect powers.
Therefore, 999 will be the 959th term in the sequence. (This is because 1000 itself is actually a perfect power; if it wasn't, 1000 would be the 959th term.)
Thus, 1001 will be the 960th term.
The next perfect power larger than 1000 is 1024.
Thus, 1023 will be the 982nd term and 1025 will be the 983rd term.
The next perfect power larger than 1024 is 1089.
Therefore, the 1042 will be the 1000th term.
The sum of the squares of the digits of 1042 is $1^2 + 0^2 + 4^2 + 2^2 = 21$.

22. The average of four different positive whole numbers is 4. If the difference between the largest and smallest of these numbers is as large as possible, what is the average of the other two numbers?
(A) $1\frac{1}{2}$ (B) $2\frac{1}{2}$ (C) 4 (D) 5 (E) 2

Source: 2007 Gauss Grade 7 #22

Primary Topics: Number Sense

Secondary Topics: Averages

Answer: B

Solution:

Since the average of four numbers is 4, their sum is $4 \times 4 = 16$.
For the difference between the largest and smallest of these numbers to be as large as possible, we would like one of the numbers to be as small as possible (so equal to 1) and the other (call it B for big) to be as large as possible.
Since one of the numbers is 1, the sum of the other three numbers is $16 - 1 = 15$.
For the B to be as large as possible, we must make the remaining two numbers (which must be different and not equal to 1) as small as possible. So these other two numbers must be equal to 2 and 3, which would make B equal to $15 - 2 - 3 = 10$.
So the average of these other two numbers is $\frac{2+3}{2} = \frac{5}{2}$ or $2\frac{1}{2}$.

23. A *Fano table* is a table with three columns where

- each entry is an integer taken from the list $1, 2, 3, \ldots, n$, and
- each row contains three different integers, and
- for each possible pair of distinct integers from the list $1, 2, 3, \ldots, n$, there is exactly one row that contains both of these integers.

The number of rows in the table will depend on the value of n. For example, the table shown is a Fano table with $n = 7$. (Notice that 2 and 6 appear in the same row only once, as does every other possible pair of the numbers $1, 2, 3, 4, 5, 6, 7$.) For how many values of n with $3 \leq n \leq 12$ can a Fano table be created?

1	2	4
2	3	5
3	4	6
4	5	7
5	6	1
6	7	2
7	1	3

(A) 2 (B) 3 (C) 5 (D) 6 (E) 7

Source: 2011 Cayley Grade 10 #23

Primary Topics: Counting and Probability

Secondary Topics: Counting | Operations

Answer: B

Solution:

First, we calculate the number of pairs that can be formed from the integers from 1 to n.
One way to form a pair is to choose one number to be the first item of the pair (n choices) and then a different number to be the second item of the pair ($n-1$ choices).
There are $n(n-1)$ ways to choose these two items in this way.
But this counts each pair twice; for example, we could choose 1 then 3 and we could also choose 3 then 1.
So we have double-counted the pairs, meaning that there are $\frac{1}{2}n(n-1)$ pairs that can be formed.
Next, we examine the number of rows in the table.
Since each row has three entries, then each row includes three pairs (first and second numbers, first and third numbers, second and third numbers).
Suppose that the completed table has r rows.
Then the total number of pairs in the table is $3r$.
Since each pair of the numbers from 1 to n appears exactly once in the table and the total number of pairs from these numbers is $\frac{1}{2}n(n-1)$, then $3r = \frac{1}{2}n(n-1)$, which tells us that $\frac{1}{2}n(n-1)$ must be divisible by 3, since $3r$ is divisible by 3.
We make a table listing the possible values of n and the corresponding values of $\frac{1}{2}n(n-1)$:

n	3	4	5	6	7	8	9	10	11	12
$\frac{1}{2}n(n-1)$	3	6	10	15	21	28	36	45	55	66

Since $\frac{1}{2}n(n-1)$ must be divisible by 3, then the possible values of n are 3, 4, 6, 7, 9, 10, and 12.
Next, consider a fixed number m from the list 1 to n.
In each row that m appears, it will belong to 2 pairs (one with each of the other two numbers in its row).
If the number m appears in s rows, then it will belong to $2s$ pairs.
Therefore, each number m must belong to an even number of pairs.
But each number m from the list of integers from 1 to n must appear in $n-1$ pairs (one with each other number in the list), so $n-1$ must be even, and so n is odd.
Therefore, the possible values of n are 3, 7, 9.
Finally, we must verify that we can create a Fano table for each of these values of n. We are given the Fano table for $n = 7$.
Since the total number of pairs when $n = 3$ is 3 and when $n = 9$ is 36, then a Fano table for $n = 3$ will have $3 \div 3 = 1$ row and a Fano table for $n = 9$ will have $36 \div 3 = 12$ rows.
For $n = 3$ and $n = 9$, possible tables are shown below:

1	2	3

1	2	3
1	4	5
1	6	7
1	8	9
2	4	7
2	5	8
2	6	9
3	4	9
3	5	6
3	7	8
4	6	8
5	7	9

In total, there are 3 values of n in this range for which a Fano table can be created.

24. A bicycle at Store P costs \$200. The regular price of the same bicycle at Store Q is 15% more than it is at Store P. The bicycle is on sale at Store Q for 10% off of the regular price. What is the sale price of the bicycle at Store Q?
(A) \$230.00 (B) \$201.50 (C) \$199.00 (D) \$207.00 (E) \$210.00

Source: 2014 Gauss Grade 7 #21

Primary Topics: Algebra and Equations

Secondary Topics: Percentages | Decimals

Answer: D

Solution:

At Store Q, the bicycle's regular price is 15% more than the price at Store P, or 15% more than \$200.
Since 15% of 200 is $\frac{15}{100} \times 200 = 0.15 \times 200 = 30$, then 15% more than \$200 is $\$200 + \$30$ or~$\$230$.
This bicycle is on sale at Store Q for 10% off of the regular price, \$230.
Since 10% of 230 is $\frac{10}{100} \times 230 = 0.10 \times 230 = 23$, then 10% off of \$230 is $\$230 - \$23$ or $\$207$.
The sale price of the bicycle at Store Q is \$207.

25. In her last basketball game, Jackie scored 36 points. These points raised the average (mean) number of points that she scored per game from 20 to 21. To raise this average to 22 points, how many points must Jackie score in her next game?
(A) 38 (B) 22 (C) 23 (D) 36 (E) 37

Source: 2016 Cayley Grade 10 #21

Primary Topics: Algebra and Equations

Secondary Topics: Averages

Answer: A

Solution:

Suppose that Jackie had played n games before her last game.
Since she scored an average of 20 points per game over these n games, then she scored $20n$ points over these n games.
In her last game, she scored 36 points and so she has now scored $20n + 36$ points in total.
But, after her last game, she has now played $n + 1$ games and has an average of 21 points scored per game.
Therefore, we can also say that her total number of points scored is $21(n + 1)$.
Thus, $21(n + 1) = 20n + 36$ or $21n + 21 = 20n + 36$ and so $n = 15$.
This tells us that after 16 games, Jackie has scored $20(15) + 36 = 336$ points.
For her average to be 22 points per game after 17 games, she must have scored a total of $17 \cdot 22 = 374$ points.
This would mean that she must score $374 - 336 = 38$ points in her next game.

26. There are n students in the math club at Scoins Secondary School. When Mrs. Fryer tries to put the n students in groups of 4, there is one group with fewer than 4 students, but all of the other groups are complete. When she tries to put the n students in groups of 3, there are 3 more complete groups than there were with groups of 4, and there is again exactly one group that is not complete. When she tries to put the n students in groups of 2, there are 5 more complete groups than there were with groups of 3, and there is again exactly one group that is not complete. The sum of the digits of the integer equal to $n^2 - n$ is
(A) 11 (B) 12 (C) 20 (D) 13 (E) 10

Source: 2016 Pascal Grade 9 #22

Primary Topics: Number Sense

Secondary Topics: Divisibility

Answer: B

Solution:

Solution 1
Suppose that, when the n students are put in groups of 2, there are g complete groups and 1 incomplete group.
Since the students are being put in groups of 2, an incomplete group must have exactly 1 student in it.
Therefore, $n = 2g + 1$.
Since the number of complete groups of 2 is 5 more than the number of complete groups of 3, then there were $g - 5$ complete groups of 3.
Since there was still an incomplete group, this incomplete group must have had exactly 1 or 2 students in it.
Therefore, $n = 3(g - 5) + 1$ or $n = 3(g - 5) + 2$.
If $n = 2g + 1$ and $n = 3(g - 5) + 1$, then $2g + 1 = 3(g - 5) + 1$ or $2g + 1 = 3g - 14$ and so $g = 15$.
In this case, $n = 2g + 1 = 31$ and there were 15 complete groups of 2 and 10 complete groups of 3.
If $n = 2g + 1$ and $n = 3(g - 5) + 2$, then $2g + 1 = 3(g - 5) + 2$ or $2g + 1 = 3g - 13$ and so $g = 14$.
In this case, $n = 2g + 1 = 29$ and there were 14 complete groups of 2 and 9 complete groups of 3.
If $n = 31$, dividing the students into groups of 4 would give 7 complete groups of 4 and 1 incomplete group.
If $n = 29$, dividing the students into groups of 4 would give 7 complete groups of 4 and 1 incomplete group.
Since the difference between the number of complete groups of 3 and the number of complete groups of 4 is given to be 3, then it must be the case that $n = 31$.
In this case, $n^2 - n = 31^2 - 31 = 930$; the sum of the digits of $n^2 - n$ is 12.

Solution 2
Since the n students cannot be divided exactly into groups of 2, 3 or 4, then n is not a multiple of 2, 3 or 4.
The first few integers larger than 1 that are not divisible by 2, 3 or 4 are 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, and 35.
In each case, we determine the number of complete groups of each size:

n	5	7	11	13	17	19	23	25	29	31	35
# of complete groups of 2	2	3	5	6	8	9	11	12	14	15	17
# of complete groups of 3	1	2	3	4	5	6	7	8	9	10	11
# of complete groups of 4	1	1	2	3	4	4	5	6	7	7	8

Since the number of complete groups of 2 is 5 more than the number of complete groups of 3 which is 3 more than the number of complete groups of 4, then of these possibilities, $n = 31$ works.
In this case, $n^2 - n = 31^2 - 31 = 930$; the sum of the digits of $n^2 - n$ is 12.
(Since the problem is a multiple choice problem and we have found a value of n that satisfies the given conditions and for which an answer is present, then this answer must be correct. Solution 1 shows why $n = 31$ is the only value of n that satisfies the given conditions.)

27. A positive integer n with $n \geq 3$ is called a *Nella number* if there exists a positive integer x with $x < n$ and there exists a positive integer m such that

- m is not divisible by x or by $x+1$, and
- m is divisible by every other positive integer between 1 and n inclusive.

For example, $n = 7$ is a Nella number. How many Nella numbers n are there with $50 \leq n \leq 2017$?

(A) 393 (B) 394 (C) 395 (D) 396 (E) 397

Source: 2017 Cayley Grade 10 #25

Primary Topics: Number Sense | Counting and Probability

Secondary Topics: Divisibility

Answer: A

Solution:

We proceed using several steps.
Step 1: Least common multiples
For each positive integer $n \geq 3$ and positive integer $x < n$, we define $L(n,x)$ to be the least common multiple of the $n-2$ integers $1, 2, 3, \ldots, x-2, x-1, x+2, x+3, \ldots, n-1, n$.
One way to calculate the least common multiple of a list of integers is to determine the prime factorization of each of the integers in the list and then to create the product of the largest of each of the prime powers that occurs among the integers in the list.
For example, if $n = 9$ and $x = 6$, then $L(9,6)$ is the least common multiple of the five integers $1, 2, 3, 4, 5, 8, 9$.
The prime factorizations of the integers larger than 1 in this list are $2 = 2^1$, $3 = 3^1$, $4 = 2^2$, $5 = 5^1$, $8 = 2^3$, $9 = 3^2$ and so $L(9,6) = 2^3 3^2 5^1 = 360$. In this case, $L(9,6)$ is not divisible by $x+1 = 7$ but is divisible by $x = 6$.
We note that, for any list of integers, a common multiple, m, of all of the integers in this list is always itself a multiple of their least common multiple l. This is because if p is a prime number and a is a positive integer for which p^a is a factor of l, then there must be an integer in the list that is a multiple of p^a. For m to be a common multiple of every number in the list, p^a must also be a factor of m. Since this is true for every prime power p^a that is a factor of l, then m is a multiple of l.
Step 2: Connection between m and $L(n,x)$
We show that n is a Nella number with corresponding x exactly when $L(n,x)$ is not divisible by x or $x+1$.
Suppose that n is a Nella number with corresponding x and m.
Then m must be divisible by each of $1, 2, 3, \ldots, x-2, x-1, x+2, x+3, \ldots, n-1, n$.
From Step 1, m is a multiple of $L(n,x)$.
Since m is a multiple of $L(n,x)$ and m is not divisible by x or $x+1$, then $L(n,x)$ is not either.
Since $L(n,x)$ is divisible by every other positive integer from between 1 and n, inclusive, then $L(n,x)$ satisfies the required conditions for m in the definition of a Nella number.
Also, if n and x are positive integers with $n \geq 3$ and $x < n$ and $L(n,x)$ has the two required conditions in the definition of a Nella number, then n is indeed a Nella number.
Putting this together, n is a Nella number with corresponding x exactly when $L(n,x)$ is not divisible by x or $x+1$.
Step 3: Re-statement of problem
Based on Step 2, we now want to find all positive integers n with $50 \leq n \leq 2017$ for which there exists a positive integer x with $x < n$ with the property that $L(n,x)$ is not divisible by x and $x+1$.
Step 4: If n is a Nella number with corresponding x, then x and $x+1$ are both prime powers
Suppose that n is a Nella number with corresponding x.
Further, suppose that x is not a prime power.
Then $x = p^a b$ for some prime number p, positive integer a and positive integer $b > 1$ that is not divisible by p.
In this case, $p^a < x$ and $b < x$ and so p^a and b are both in the list $1, 2, 3, \ldots, x-2, x-1$, which means that $L(n,x)$ is a multiple of both p^a and b and so is a multiple of $p^a b = x$. (It is important here that p^a and b have no common prime factors.) This is a contradiction.
This means that if n is a Nella number, then x is a prime power.
Similarly, $x+1$ must also be a prime power. To see this, we use the same argument with the additional observation that, because $x+1$ and x are consecutive, they cannot have any common divisor larger than 1 and so if $x+1 = p^c d$, then x cannot equal p^c or d and therefore both p^c and d are indeed in the list $1, 2, 3, \ldots, x-1$.
Step 5: Further analysis of x and $x+1$
Suppose that n is a Nella number with corresponding x.
From Step 4, both x and $x+1$ are prime powers.
Since x and $x+1$ are consecutive, then one is even and one is odd.
In other words, one of x and $x+1$ is a power of 2 and the other is a power of an odd prime.
Furthermore, we know that $L(n,x)$ is not divisible by x or $x+1$.
This means that the list $x+2, x+3, \ldots, n-1, n$ cannot contain a multiple of x or $x+1$.
This means that $x < n < 2x$ and $x+1 \leq n < 2(x+1)$, because the "next" multiple of x is $2x$ and the next multiple of $x+1$ is $2(x+1)$.
Since one of x and $x+1$ is a power of 2, then 2 times this power of 2 (that is, $2x$ or $2(x+1)$) is the next power of 2, and so the inequalities tell us that n is smaller than the next power of 2, and so x or

$x+1$ has to be the largest power of 2 less than (or less than or equal to, respectively) n.

Step 6: Second re-statement of problem

We want to find all positive integers n with $50 \le n \le 2017$ for which there exists a positive integer x with $x < n$ with the property that $L(n,x)$ is not divisible by x and $x+1$.

From Step 5, we know that either x is the largest power of 2 less than n or $x+1$ is the largest power of 2 less than or equal to n.

This means that we want to find all positive integers n with $50 \le n \le 2017$ for which at least one of the following two statements is true for some positive integer $x < n$:

- If x is the largest power of 2 less than n, then $x+1$ is also a prime power.
- If $x+1$ is the largest power of 2 less than or equal to n, then x is also a prime power.

Step 7: Determining Nella numbers

We make two separate tables, one where $x < n$ is a power of 2 and one where $x+1 \le n$ is a power of 2. In each case, we determine whether $x+1$ or x is also a prime power.

Range of n	**Largest power of 2 less than n**	x	$x+1$	**Prime power?**	**Nella?**
$50 \le n \le 64$	32	32	33	No ($33 = 3 \cdot 11$)	No
$65 \le n \le 128$	64	64	65	No ($65 = 5 \cdot 13$)	No
$129 \le n \le 256$	128	128	129	No ($129 = 3 \cdot 43$)	No
$257 \le n \le 512$	256	256	257	Yes	See below
$513 \le n \le 1024$	512	512	513	No ($513 = 3 \cdot 171$)	No
$1025 \le n \le 2017$	1024	1024	1025	No ($1025 = 5 \cdot 205$)	No

Note that 257 is a prime number since it is not divisible by any prime number less than $\sqrt{257} \approx 16.03$. (These primes are $2, 3, 5, 7, 11, 13$.)

For each n with $257 \le n \le 511$, $L(n, 256)$ is not divisible by 256 or 257, so each of these n (there are $511 - 257 + 1 = 255$ of them) is a Nella number.

For $n = 512$, $L(n, 256)$ is divisible by 256 (since 512 is divisible by 256), so $n = 512$ is not a Nella number as this is the only possible candidate for x in this case.

Range of n	**Largest power of 2 at most n**	$x+1$	x	**Prime power?**	**Nella?**
$50 \le n \le 63$	32	32	31	Yes	See below
$64 \le n \le 127$	64	64	63	No ($63 = 3 \cdot 21$)	No
$128 \le n \le 255$	128	128	127	Yes	See below
$256 \le n \le 511$	256	256	255	No ($255 = 5 \cdot 51$)	No
$512 \le n \le 1023$	512	512	511	No ($511 = 7 \cdot 73$)	No
$1024 \le n \le 2017$	1024	1024	1023	No ($1023 = 3 \cdot 341$)	No

Note that 31 and 127 are prime.

For each n with $50 \le n \le 61$, $L(n, 31)$ is not divisible by 31 or 32, so each of these n (there are $61 - 50 + 1 = 12$ of them) is a Nella number.

For $n = 62, 63$, $L(n, 31)$ is divisible by 31 (since 62 is divisible by 31), so neither is a Nella number as this is the only possible candidate for x in this case.

For each n with $128 \le n \le 253$, $L(n, 127)$ is not divisible by 127 or 128, so each of these n (there are $253 - 128 + 1 = 126$ of them) is a Nella number.

For $n = 254, 255$, $L(n, 127)$ is divisible by 127 (since 254 is divisible by 127), so neither is a Nella number as this is the only possible candidate for x in this case.

Step 8: Final tally

From the work above, there are $255 + 12 + 126 = 393$ Nella numbers n with $50 \le n \le 2017$.

If x is the largest power of 2 less than n, then $x+1$ is also a prime power.

If $x+1$ is the largest power of 2 less than or equal to n, then x is also a prime power.

28. In the multiplication shown, each of P, Q, R, S, and T is a digit. The value of $P+Q+R+S+T$ is
(A) 14 (B) 20 (C) 16
(D) 17 (E) 13

$$\begin{array}{r} P\;Q\;R\;S\;T\;4 \\ \times \qquad\qquad 4 \\ \hline 4\;P\;Q\;R\;S\;T \end{array}$$

Source: 2019 Cayley Grade 10 #21

Primary Topics: Number Sense

Secondary Topics: Digits | Equations Solving | Operations

Answer: A

Solution:

Solution 1
We start with the ones digits.
Since $4 \times 4 = 16$, then $T = 6$ and we carry 1 to the tens column.
Looking at the tens column, since $4 \times 6 + 1 = 25$, then $S = 5$ and we carry 2 to the hundreds column.
Looking at the hundreds column, since $4 \times 5 + 2 = 22$, then $R = 2$ and we carry 2 to the thousands column.
Looking at the thousands column, since $4 \times 2 + 2 = 10$, then $Q = 0$ and we carry 1 to the ten thousands column.
Looking at the ten thousands column, since $4 \times 0 + 1 = 1$, then $P = 1$ and we carry 0 to the hundred thousands column.
Looking at the hundred thousands column, $4 \times 1 + 0 = 4$, as expected.
This gives the following completed multiplication:

$$\begin{array}{r} 1\;0\;2\;5\;6\;4 \\ \times \qquad\qquad 4 \\ \hline 4\;1\;0\;2\;5\;6 \end{array}$$

Finally, $P+Q+R+S+T = 1+0+2+5+6 = 14$.

Solution 2
Let x be the five-digit integer with digits $PQRST$.
This means that $PQRST0 = 10x$ and so $PQRST4 = 10x + 4$.
Also, $4PQRST = 400\,000 + PQRST = 400\,0000 + x$.
From the given multiplication, $4(10x + 4) = 400\,000 + x$ which gives $40x + 16 = 400\,000 + x$ or $39x = 399\,984$.
Thus, $x = \dfrac{399\,984}{39} = 10\,256$.
Since $PQRST = 10\,256$, then $P+Q+R+S+T = 1+0+2+5+6 = 14$.

29. Consider positive integers $a \leq b \leq c \leq d \leq e$. There are N lists a, b, c, d, e with a mean of 2023 and a median of 2023, in which the integer 2023 appears more than once, and in which no other integer appears more than once. What is the sum of the digits of N?

Source: 2023 Pascal Grade 9 #25

Primary Topics: Algebra and Equations

Secondary Topics: Inequalities | Averages | Decimals

Answer: 28

Solution:

Since the median of the list a, b, c, d, e is 2023 and $a \le b \le c \le d \le e$, then $c = 2023$. Since 2023 appears more than once in the list, then it appears 5, 4, 3, or 2 times.
Case 1: 2023 appears 5 times.
Here, the list is 2023, 2023, 2023, 2023, 2023. There is 1 such list.
Case 2: 2023 appears 4 times.
Here, the list would be 2023, 2023, 2023, 2023, x where x is either less than or greater than 2023. Since the mean of the list is 2023, the sum of the numbers in the list is 5×2023, which means that $x = 5 \times 2023 - 4 \times 2023 = 2023$, which is a contradiction. There are 0 lists in this case.
Case 3: 2023 appears 3 times.
Here, the list is a, b, 2023, 2023, 2023 (with $a < b < 2023$) or a, 2023, 2023, 2023, e (with $a < 2023 < e$), or 2023, 2023, 2023, d, e (with $2023 < d < e$). In the first case, the mean of the list is less than 2023, since the sum of the numbers will be less than 5×2023. In the third case, the mean of the list is greater than 2023, since the sum of the numbers will be greater than 5×2023. So we need to consider the list a, 2023, 2023, 2023, e with $a < 2023 < e$. Since the mean of this list is 2023, then the sum of the five numbers is 5×2023, which means that $a + e = 2 \times 2023$. Since a is a positive integer, then $1 \le a \le 2022$. For each such value of a, there is a corresponding value of e equal to $4046 - a$, which is indeed greater than 2023. Since there are 2022 choices for a, there are 2022 lists in this case.
Case 4A: 2023 appears 2 times; $c = d = 2023$.
(We note that if 2023 appears 2 times, then since $c = 2023$ and $a \le b \le c \le d \le e$, we either have $c = d = 2023$ or $b = c = 2023$.) Here, the list is a, b, 2023, 2023, e with $1 \le a < b < 2023 < e$. This list has median 2023 and no other integer appears more than once. Thus, it still needs to satisfy the condition about the mean. For this to be the case, the sum of its numbers equals 5×2023, which means that $a + b + e = 3 \times 2023 = 6069$. Every pair of values for a and b with $1 \le a < b < 2023$ will give such a list by defining $e = 6069 - a - b$. (We note that since $a<b<2023$ we will indeed have $e > 2023$.) If $a = 1$, there are 2021 possible values for b, namely $2 \le b \le 2022$. If $a = 2$, there are 2020 possible values for b, namely $3 \le b \le 2022$. Each time we increase a by 1, there will be 1 fewer possible value for b, until $a = 2021$ and $b = 2022$ (only one value). Therefore, the number of pairs of values for a and b in this case is

$$2021 + 2020 + \cdots + 2 + 1 = \frac{1}{2} \times 2021 \times 2022 = 2021 \times 1011$$

This is also the number of lists in this case.
Case 4B: 2023 appears 2 times; $b = c = 2023$.
Here, the list is a, 2023, 2023, d, e with $1 \le a < 2023 < d < e$. This list has median 2023 and no other integer appears more than once. Thus, it still needs to satisfy the condition about the mean. For this to be the case, the sum of its numbers equals 5×2023, which means that $a + d + e = 3 \times 2023 = 6069$. If $d = 2024$, then $a + e = 4045$. Since $1 \le a \le 2022$ and $2025 \le e$, we could have $e = 2025$ and $a = 2020$, or $e = 2026$ and $a = 1019$, and so on. There are 2020 such pairs, since once a reaches 1, there are no more possibilities. If $d = 2025$, then $a + e = 4044$. Since $1 \le a \le 2022$ and $2026 \le e$, we could have $e = 2026$ and $a = 2018$, or $e = 2027$ and $a = 1017$, and so on. There are 2018 such pairs. As d increases successively by 1, the sum $a + e$ decreases by 1 and the minimum value for e increases by 1, which means that the maximum value for a decreases by 2, which means that the number of pairs of values for a and e decreases by 2. This continues until we reach $d = 3033$ at which point there are 2 pairs for a and e. Therefore, the number of pairs of values for a and e in this case is

$$2020 + 2018 + 2016 + \cdots + 4 + 2$$

which is equal to

$$2 \times (1 + 2 + \cdots + 1008 + 1009 + 1010)$$

which is in turn equal to $2 \times \frac{1}{2} \times 1010 \times 1011$ which equals1010×1011.
Combining all of the cases, the total number of lists a, b, c, d, e is

$$N = 1 + 2022 + 2021 \times 1011 + 1010 \times 1011 = 1 + 1011 \times (2 + 2021 + 1010) = 1 + 1011 \times 3033$$

and so $N = 3\,066\,364$. The sum of the digits of N is $3 + 0 + 6 + 6 + 3 + 6 + 4$ or 28.

30. Three circles have radii 1 cm, 5 cm, and x cm. If the mean (average) area of the three circles is $30\pi cm^2$, the value of x is

(A) 64 (B) 5 (C) 24 (D) 8 (E) 2

Source: 2025 Gauss Grade 7 #21

Primary Topics: Geometry and Measurement

Secondary Topics: Circles | Measurement | Averages

Answer: D

Solution:

A circle with radius 1 cm has area $\pi \times (1\text{ cm})^2 = \pi\text{ cm}^2$.
A circle with radius 5 cm has area $\pi \times (5\text{ cm})^2 = 25\pi\text{ cm}^2$.
A circle with radius x cm has area $\pi \times (x\text{ cm})^2 = x^2\pi\text{ cm}^2$.
Since the mean area of the three circles is $30\pi\text{ cm}^2$, then the sum of the areas of the three circles is $3 \times 30\pi\text{ cm}^2 = 90\pi\text{ cm}^2$.
Since $\pi + 25\pi + x^2\pi = 26\pi + x^2\pi$, we get $26\pi + x^2\pi = 90\pi$ or $x^2\pi = 64\pi$ or $x^2 = 64$, and so $x = 8$ (since $x > 0$).